

GENERALIZED COSECANT NUMBERS AND THE HURWITZ ZETA FUNCTION

VICTOR KOWALENKO

ABSTRACT. This announcement paper summarises recent development concerning the generalized cosecant numbers $c_{\rho,k}$, which represent the coefficients of the power series expansion for the important fundamental function $z^\rho/\sin^\rho z$. These coefficients are obtained for all, including complex, values of ρ via the partition method for a power series expansion, which is more versatile than the standard Taylor series approach, but yields the same results as the latter when both can be applied, though in a different form. Surprisingly, the generalized cosecant numbers are polynomials in ρ of degree k , where k is the power of z . General formulas for the coefficients of the highest order terms in the generalized cosecant numbers are presented. It is then shown how the generalized cosecant numbers are related to the specific symmetric polynomials from summing over quadratic powers of integers. Consequently, integral values of the Hurwitz zeta function for even powers are expressed for the first time ever in terms of ratios of the generalized cosecant numbers.

The cosecant numbers, c_k , are defined in Ref. [1] as the rational coefficients of the power series expansion for cosecant, in particular by

$$\csc z = \frac{1}{\sin z} \equiv \sum_{k=0}^{\infty} c_k z^{2k-1}. \quad (1)$$

Via the partition method for a power series expansion a general formula for them is derived in terms of all the integer partitions summing to k , which is

$$c_k = (-1)^k \sum_{\substack{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k \\ \sum_{i=1}^k i \lambda_i = k}}^{k, \lfloor k/2 \rfloor, \lfloor k/3 \rfloor, \dots, 1} (-1)^{N_k} N_k! \prod_{i=1}^k \left(\frac{1}{(2i+1)!} \right)^{\lambda_i} \frac{1}{\lambda_i!}. \quad (2)$$

In this result λ_i represents the multiplicity or the number of occurrences of each part i in the partitions, while the sum of the multiplicities of each partition is represented by N_k , i.e. $N_k = \sum_{i=1}^k \lambda_i$. For the partitions summing to k , the multiplicity of a part i ranges from zero to $\lfloor k/i \rfloor$, where $\lfloor x \rfloor$ denotes the floor function or the greatest integer less than or equal to x . Generally, when k is large, most of multiplicities for a partition vanish as we shall see shortly. Using (2) one finds that $c_0 = 1$, $c_1 = 1/6$, $c_2 = 7/360$, $c_4 = 31/3 \cdot 7!$ and so on. In addition, the reader should take note that an equivalence symbol has been introduced into (1) because the power series expansion on the rhs is divergent for $|z| \geq \pi$, while the lhs is always defined. For these values the rhs must be regularized in the manner described in Refs. [1]-[7]. Since the power series is convergent for the other values of z , it is permissible to replace the equivalence symbol by an equals sign. In addition, in Ref. [1] it is proved that the cosecant numbers can be expressed in terms of the Riemann zeta

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Partition	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	N_k
$\{6\}$						1	1
$\{5, 1\}$	1				1		2
$\{4, 2\}$		1		1			2
$\{4, 1, 1\}$	2			1			3
$\{3, 3\}$			2				2
$\{3, 2, 1\}$	1	1	1				3
$\{3, 1, 1, 1\}$	3		1				4
$\{2, 2, 2\}$		3					3
$\{2, 2, 1, 1\}$	2	2					4
$\{2, 1, 1, 1, 1\}$	4	1					5
$\{1, 1, 1, 1, 1, 1\}$	6						6

TABLE 1. Multiplicities of the eleven partitions with parts summing to 6

function as

$$c_k = 2(2 - 2^{-2k}) \frac{\zeta(2k)}{\pi^{2k}}, \quad (3)$$

where $\zeta(2k)$ represents the Riemann zeta function. Therefore, (2) is effectively another method of determining even integer values of this major function in number theory.

Ref. [1] not only presents numerous applications of cosecant numbers, but also demonstrates how they are related to other numbers such as the secant numbers and, more importantly, how the sets of the resulting numbers can be generalized or extended by introducing an arbitrary power ρ to their generating function. Specifically, the generalized cosecant numbers are given by

$$\csc^\rho z = \frac{1}{\sin^\rho z} \equiv \sum_{k=0}^{\infty} c_{\rho,k} z^{2k-\rho}, \quad (4)$$

where

$$c_{\rho,k} = (-1)^k \sum_{\substack{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k \\ \sum_{i=1}^k i \lambda_i = k}}^{k, \lfloor k/2 \rfloor, \lfloor k/3 \rfloor, \dots, 1} (-1)^{N_k(\rho)} \prod_{i=1}^k \left(\frac{1}{(2i+1)!} \right)^{\lambda_i} \frac{1}{\lambda_i!}. \quad (5)$$

In (5) $(\rho)_{N_k}$ denotes the Pochhammer notation for $\Gamma(\rho + N_k)/\Gamma(\rho)$, where $\Gamma(x)$ represents the gamma function.

To calculate the generalized cosecant number $c_{\rho,k}$ via (5), we need to determine the specific contribution made by each integer partition that sums to k . For example, if we wish to evaluate $c_{\rho,6}$ we require all the contributions made from all eleven partitions that sum to 6. These are listed in the first column of Table 1. Each part in a partition is assigned a specific value, which depends on the function under consideration. In the case of $x^\rho/\sin^\rho x$, the part i is assigned a value of $(-1)^{i+1}/(2i+1)!$. In addition, since each part occurs λ_i times in a partition, then we need to multiply λ_i values or calculate $(-1)^{(i+1)\lambda_i}/((2i+1)!)^{\lambda_i}$. The second column in Table 1 displays the multiplicities of the parts in all the partitions summing to 6, while the third column presents N_k for each partition. Thus, we observe that most of the multiplicities vanish as stated earlier.

Associated with each partition is a multinomial factor that is determined by taking the factorial of the total number of parts in a partition and dividing by the factorials of all the multiplicities. For the partition $\{2, 1, 1, 1, 1\}$ in Table 1, we have $\lambda_1 = 4$ and $\lambda_2 = 1$ with all other multiplicities equal to zero. Hence, the multinomial factor for this partition is

$5!/(4!1!) = 5$. When the function is accompanied by an arbitrary power, say ρ , a further modification must be made. Each partition is then multiplied by the Pochhammer factor of $\Gamma(N_k + \rho)/\Gamma(\rho)$ divided by $N_k!$. That is, for each partition we must include the extra factor of $(\rho)_{N_k}/N_k!$. For $\rho = 1$, this simply yields unity and thus, the multinomial factor remains unaffected. Consequently, (5) reduces to (2) for $\rho = 1$. For $\rho = 2$ we obtain the cosecant-squared numbers as given in Theorem 5 of Ref. [1], which are given by

$$\begin{aligned} c_{2,k} &= (-1)^k \sum_{\substack{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k \\ \sum_{i=1}^k i \lambda_i = k}}^{k, [k/2], [k/3], \dots, 1} (-1)^{N_k} (2)_{N_k} \prod_{i=1}^k \left(\frac{1}{(2i+1)!} \right)^{\lambda_i} \frac{1}{\lambda_i!} \\ &= \frac{(2k-1)}{(1-2^{1-2k})} c_k. \end{aligned} \quad (6)$$

In the above result one can replace $(2)_{N_k}$ by $(N_k+1)(1)_{N_k}$. Then the part involving unity yields c_k . Hence we are left with (2) again except the $N_k!$ is multiplied by N_k . On the other hand, if $\rho = -1$, then the coefficients are simply equal to the power series expansion for $\sin z$ divided by z . Therefore, we find that

$$\sum_{\substack{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k \\ \sum_{i=1}^k i \lambda_i = k}}^{k, [k/2], [k/3], \dots, 1} (-1)^{N_k} (\rho)_{N_k} \prod_{i=1}^k \left(\frac{1}{(2i+1)!} \right)^{\lambda_i} \frac{1}{\lambda_i!} = \frac{1}{(2k+1)!}. \quad (7)$$

Hence, we have an expression for the reciprocal of $(2k+1)!$ in terms of a sum over partitions summing to k .

If we examine (5) more closely, then we see that the product deals with the calculating the contribution made by each partition based on the values of the multiplicities, while the sum refers to all partitions summing to k . Hence the sum covers the range of values for each multiplicity. For example, λ_1 attains a maximum value of k , which corresponds the partition with k ones, while λ_2 attains a maximum value of $[k/2]$, which corresponds to the partition with $[k/2]$ twos in it. For odd values of k , the partition with $[k/2]$ twos also possesses a one, i.e. $\lambda_1 = 1$. Thus, it can be seen that the maximum value of λ_i is always $[k/i]$, which becomes the upper limit for each multiplicity in both (2) and (5). Moreover, each partition in the sums must satisfy the constraint, $\sum_{i=1}^k i \lambda_i = k$.

As an example, consider the evaluation of $c_{\rho,6}$. According to Table 1 there are eleven partitions summing to 6. Therefore, we need to determine eleven contributions in the sum over the partitions. By applying the steps given above to (5), we find that the contributions from the partitions in the same order as the table are

$$\begin{aligned} c_{\rho,6} &= -(\rho)_1 \frac{1}{13!} + \frac{(\rho)_2}{2!} \frac{2!}{1! \cdot 1!} \frac{1}{3! \cdot 11!} + \frac{(\rho)_2}{2!} \frac{2!}{1! \cdot 1!} \frac{1}{5! \cdot 9!} - \frac{(\rho)_3}{3!} \frac{3!}{1! \cdot 2!} \frac{1}{3!^2 \cdot 9!} \\ &+ \frac{(\rho)_2}{2!} \frac{2!}{2!} \frac{1}{(7!)^2} - \frac{(\rho)_3}{3!} \frac{3!}{1! \cdot 1! \cdot 1!} \frac{1}{3! \cdot 5! \cdot 7!} + \frac{(\rho)_4}{4!} \frac{4!}{1! \cdot 3!} \frac{1}{(3!)^3 \cdot 7!} - \frac{(\rho)_3}{3!} \frac{3!}{3!} \frac{1}{(5!)^3} \\ &+ \frac{(\rho)_4}{4!} \frac{4!}{2! \cdot 2!} \frac{1}{(3!)^2 \cdot (5!)^2} - \frac{(\rho)_5}{5!} \frac{5!}{4! \cdot 1!} \frac{1}{(3!)^4 \cdot 5!} + \frac{(\rho)_6}{6!} \frac{6!}{6!} \frac{1}{(3!)^6}. \end{aligned} \quad (8)$$

The interesting property of the above result is that when the number of parts N_k given by the subscript on the Pochhammer terms is even, the contribution from the partition is positive while if it is odd, then the contribution is negative. This applies to all even values of k . However, if k is odd, then the reverse applies. The contributions with an odd number of parts are positive, while those from an even number of parts are negative.

Furthermore, by introducing (8) into Mathematica [8] and wrapping it entirely around the combination of the Simplify and Expand routines, one obtains

$$c_{\rho,6} = \frac{1}{5884534656000} \left(1061376\rho + 3327584\rho^2 + 4252248\rho^3 + 2862860\rho^4 + 1051050\rho^5 + 175175\rho^6 \right). \quad (9)$$

Since the denominator equals $2/(9 \cdot 15!)$, we arrive at the $k = 6$ result in Table 2.

Despite the fact that the contributions alternate in sign according to whether the number of parts in each partition is even or odd, the $c_{\rho,k}$ possess only positive coefficients. From (9) we see that $c_{\rho,6}$ is a sixth order polynomial in ρ . The highest order term arises from the partition with six ones in it, which produces the term with $(\rho)_6$. Thus, it can be seen that the generalized cosecant numbers $c_{\rho,k}$ are polynomials of degree k with fixed coefficients that can be represented by $c_{\rho,k} = \sum_{i=1}^k C_{k,i} \rho^i$. General formulas for the highest order coefficients have recently been calculated in Ref. [9]. These are

$$C_{k,k} = \frac{1}{(3!)^k k!}, \quad C_{k,k-1} = \frac{1}{5 \cdot (3!)^k (k-2)!}, \quad C_{k,k-2} = \frac{(21k+17)}{175(3!)^{k+1}(k-3)!},$$

and

$$C_{k,k-3} = \frac{k^2 + 17k/7}{125(3!)^{k+1}(k-4)!}. \quad (10)$$

These results for $C_{k,k-\ell}$ become progressively more difficult as ℓ increases. For example, it has already been mentioned that $\ell = 0$ result is composed only of the contribution from the partition with k ones in it. Since $\lambda_1 = k$, (5) yields $(\rho)_k / (3!)^k k!$. Hence, the highest power of ρ in $(\rho)_k$ has a coefficient of unity, we arrive at the result for $C_{k,k}$ in (14). We also note that this term contributes to all the other powers of ρ . Therefore, while there is only one partition with $k-1$ parts, we need to include the coefficient of ρ^{k-1} from the partition with k ones in it in order to determine the general formula for $C_{k,k-1}$. For the first few values of ℓ , this is manageable, but becomes formidable with higher values. However, a pattern appears in the above results that can avoid this approach. First, the results in (9) for $C_{k,k-\ell}$ possess in their denominators $(3!)^k (k-1-\ell)!$. The numerator can be expressed as a polynomial in k of degree $\ell-1$. For $C_{k,k-3}$ one conjectures that

$$C_{k,k-3}(k) = \frac{ak^2 + bk + c}{(3!)^{k+2}(k-4)!}. \quad (11)$$

Now we let $k = 4$ in the above result and equate it to $C_{4,1}(k)$ in Table 2, which in turn equals $144/5443200$. Similarly, if we put $k = 5$ and $k = 6$ in (11) and equate these values respectively to $C_{5,2}(k)$ and $C_{6,3}(k)$ in the table, then we obtain the following set of equations:

$$16a + 4b + c = \frac{216}{175}, \quad (12)$$

$$25a + 5b + c = \frac{312}{175}, \quad (13)$$

$$36a + 6b + c = \frac{2124}{875}. \quad (14)$$

The solution to the above set of equations is $a = 6/125$, $b = 102/875$ and $c = 0$. Hence, we arrive at the result for $C_{k,k-3}(k)$ given above.

Table 2 displays the generalized cosecant numbers up to $k = 15$, which have been obtained by introducing the multiplicities of all partitions summing to k into the sum in

k	$c_{\rho,k}$
0	1
1	$\frac{1}{3!} \rho$
2	$\frac{2}{6!} (2\rho + 5\rho^2)$
3	$\frac{8}{9!} (16\rho + 42\rho^2 + 35\rho^3)$
4	$\frac{2}{3 \cdot 10!} (144\rho + 404\rho^2 + 420\rho^3 + 175\rho^4)$
5	$\frac{4}{3 \cdot 12!} (768\rho + 2288\rho^2 + 2684\rho^3 + 1540\rho^4 + 385\rho^5)$
6	$\frac{2}{9 \cdot 15!} (1061376\rho + 3327594\rho^2 + 4252248\rho^3 + 2862860\rho^4 + 1051050\rho^5 + 175175\rho^6)$
7	$\frac{1}{27 \cdot 15!} (552960\rho + 1810176\rho^2 + 2471456\rho^3 + 1849848\rho^4 + 820820\rho^5 + 210210\rho^6 + 25025\rho^7)$
8	$\frac{2}{45 \cdot 18!} (200005632\rho + 679395072\rho^2 + 978649472\rho^3 + 792548432\rho^4 + 397517120\rho^5 + 125925800\rho^6 + 23823800\rho^7 + 2127125\rho^8)$
9	$\frac{4}{81 \cdot 21!} (129369047040\rho + 453757851648\rho^2 + 683526873856\rho^3 + 589153364352\rho^4 + 323159810064\rho^5 + 117327450240\rho^6 + 27973905960\rho^7 + 4073869800\rho^8 + 282907625\rho^9)$
10	$\frac{2}{6075 \cdot 22!} (38930128699392\rho + 140441050828800\rho^2 + 219792161825280\rho^3 + 199416835425280\rho^4 + 117302530691808\rho^5 + 47005085727600\rho^6 + 12995644662000\rho^7 + 2422012593000\rho^8 + 280078548750\rho^9 + 15559919375\rho^{10})$
11	$\frac{8}{243 \cdot 25!} (494848416153600\rho + 1830317979303936\rho^2 + 2961137042841600\rho^3 + 2805729689044480\rho^4 + 1747214980192000\rho^5 + 755817391389984\rho^6 + 232489541684400\rho^7 + 50749166067600\rho^8 + 7607466867000\rho^9 + 715756291250\rho^{10} + 32534376875\rho^{11})$
12	$\frac{2}{2835 \cdot 27!} (1505662706987827200\rho + 5695207005856038912\rho^2 + 9487372599204065280\rho^3 + 9332354263294766080\rho^4 + 6096633539052376320\rho^5 + 2806128331871953088\rho^6 + 937291839756592320\rho^7 + 229239926321406000\rho^8 + 40598842049766000\rho^9 + 5005999501002500\rho^{10} + 390802935022500\rho^{11} + 14803141478125\rho^{12})$
13	$\frac{232}{81 \cdot 30!} (844922884529848320\rho + 3261358271400247296\rho^2 + 5576528334428209152\rho^3 + 5668465199488266240\rho^4 + 3858582205451484160\rho^5 + 1870620248833400064\rho^6 + 667822651436228288\rho^7 + 178292330746770240\rho^8 + 35600276746834800\rho^9 + 5225593531158000\rho^{10} + 539680243602500\rho^{11} + 35527539547500\rho^{12} + 1138703190625\rho^{13})$
14	$\frac{2}{1215 \cdot 30!} (138319015041155727360\rho + 543855095595477762048\rho^2 + 952027796641042464768\rho^3 + 996352286992030556160\rho^4 + 703040965960031795200\rho^5 + 356312537387839432192\rho^6 + 134466795172062184832\rho^7 + 38526945410311117760\rho^8 + 8436987713444690400\rho^9 + 1404048942958662000\rho^{10} + 173777038440005000\rho^{11} + 15258232341852500\rho^{12} + 858582205731250\rho^{13} + 23587423234375\rho^{14})$
15	$\frac{1088}{729 \cdot 35!} (562009739464769840087040\rho + 2247511941596311764074496\rho^2 + 4019108379306905439830016\rho^3 + 4317745925208072594259968\rho^4 + 3145163776677939429416960\rho^5 + 1656917203539032341530624\rho^6 + 655643919364420586023424\rho^7 + 199227919419039256217472\rho^8 + 46995751664475880185920\rho^9 + 8614026107092938211680\rho^{10} + 1214778349162323946000\rho^{11} + 128587452922193265000\rho^{12} + 9720180867524627500\rho^{13} + 472946705787806250\rho^{14} + 11260635852090625\rho^{15})$

TABLE 2. Generalized cosecant numbers $c_{\rho,k}$ up to $k = 15$

(2). For $k > 10$, the partition method for a power series expansion becomes laborious due to the exponential increase in the number of partitions. To circumvent this problem, a general computing methodology has been developed [9, 10], which is based on representing all the partitions summing to a specific order k by a partition tree and invoking the

bivariate recursive central partition (BRCP) algorithm. This results in general expressions for the coefficients of any power series expansion obtained from the method for a power series expansion. For example, the general symbolic form for partitions summing to 6 in the resulting power series expansion is given by

$$\begin{aligned} \text{DS}[6] := & p[6] q[1] a + p[1] p[5] q[2] a^{(2)} 2! + p[1]^{(2)} p[4] q[3] a^{(3)} 3!/2! + \\ & p[1]^{(3)} p[3] q[4] a^{(4)} 4!/3! + p[1]^{(4)} p[2] q[5] a^{(5)} 5!/4! + p[1]^{(6)} q[6] a^{(6)} + \\ & p[1]^{(2)} p[2]^{(2)} q[4] a^{(4)} 4!/(2! 2!) + p[1] p[2] p[3] q[3] a^{(3)} 3! + \\ & + p[2] p[4] q[2] a^{(2)} 2! + p[2]^{(3)} q[3] a^{(3)} + p[3]^{(2)} q[2] a^{(2)} \quad . \end{aligned}$$

Such an expression can be imported into Mathematica [8]. Then the coefficients of the inner series $p[k]$ are set equal to the assigned values of the parts. For the generalized cosecant numbers, this means we set

$$p[k_-] := (-1)^{(k+1)}/(2k+1)! \quad ,$$

while the $q[k]$, which are referred to as the coefficients of the outer series, are set equal to the coefficients of the binomial series, viz.

$$q[k_-] := \text{Pochhammer}[\rho, k]/k! \quad .$$

In the expression for $\text{DS}[6]$ we set the parameter a equal to unity, thereby obtaining $c_{\rho,6}$. On the other hand, changing $p[k]$ to

$$p[k_-] := (-1)^{(k+1)}/(2k)! \quad ,$$

will yield an entirely different set numbers known as the generalized secant numbers $d_{\rho,k}$ [1, 9]. These numbers represent the coefficients in the power series expansion for $\sec^{\rho} z$. In terms of the partition method for a power series expansion they are given by

$$d_{\rho,k} = (-1)^k \sum_{\substack{n_1, n_2, n_3, \dots, n_k=0 \\ \sum_{i=1}^k i n_i = k}}^{k, [k/2], [k/3], \dots, 1} (-1)^N (\rho)_N \prod_{i=1}^k \left(\frac{1}{(2i)!} \right)^{n_i} \frac{1}{n_i!} \quad . \quad (15)$$

Therefore, we observe that the symbolic representation for $\text{DS}[6]$ is not only general, but also powerful.

For $|\rho| \gg k$ we may use the coefficients in (14) as a means of approximating the generalized cosecant numbers. That is, they can be expressed as

$$c_{\rho,k} \stackrel{|\rho| \gg k}{\approx} \frac{\rho^k}{(3!)^k k!} + \frac{\rho^{k-1}}{5 \cdot (3!)^k (k-2)!} + \frac{(21k+17)\rho^{k-2}}{175 \cdot (3!)^{k+1} (k-3)!} + \frac{(k^2+17k/7)\rho^{k-3}}{125 \cdot (3!)^{k+1} (k-4)!} \quad . \quad (16)$$

To gain an appreciation of this approximation, let us denote the ratio of (16) to the corresponding values of $c_{\rho,k}$ in Table 2 by $\beta(\rho, k)$. Table 3 presents values of $\beta(\rho, k)$ for integer values of ρ ranging from 10 to 1000. They have been given to six decimal places with no rounding-off. From the table we see that when ρ is close to k or smaller, which occurs towards the right hand top corner, $\beta(\rho, k)$ is not close to unity, but for all other values, it is. Therefore, provided ρ is significantly greater than k , (16) represents an accurate approximation for the generalized cosecant numbers.

There is one interesting feature in the table that needs to be mentioned. If one examines the ratios when (1) $k = 10$ and $\rho = 20$ and (2) $k = 15$ and $\rho = 30$, then it is readily observed that the ratio is more accurate for the first case than in the second case despite the fact they are both good approximations. This means that as k increases, $|\rho/k|$ must also increase in order to obtain the same value of $\beta(\rho, k)$ for the lower values of k . For example, the value of $\beta(20, 10)$ is about 0.991, which in the case of $k = 15$ only is only

ρ	$k = 6$	$k = 8$	$k = 10$	$k = 12$	$k = 15$
10	0.998905	0.985655	0.929830	0.801477	0.501086
15	0.999741	0.996144	0.977941	0.925497	0.751944
20	0.999910	0.998571	0.991119	0.966957	0.870459
30	0.999981	0.999669	0.997755	0.990752	0.956671
50	0.999997	0.999951	0.999644	0.998405	0.991292
100	0.999999	0.999997	0.999974	0.999676	0.992370
1000	0.999999	0.999999	0.999999	0.999999	0.999999

TABLE 3. The ratio $\beta(\rho, k)$ for various values of ρ and k

reached when ρ is about 50. That is, in the $k = 10$ case it only takes twice the value of k to achieve the same value of $\beta(\rho, k)$ as the $k = 15$ case, which requires at least three times the value of k . This has ramifications when the relationship between the generalized cosecant numbers and the Hurwitz zeta function is presented shortly.

In another recent work [11] it has been shown that the generalized cosecant numbers are also given by

$$c_{2v,i} = 2^{2i} \frac{\Gamma(2v-2i)}{\Gamma(2v)} s(v, i), \quad i < v, \quad (17)$$

where $s(v, n)$ represents the n th elementary symmetric polynomial obtained by summing over quadratic powers or squared integers, viz. $1^2, 2^2, \dots, (v-1)^2$. That is,

$$s(v, n) = \sum_{1 \leq i_1 < i_2 < \dots < i_n < v-1} x_{i_1} x_{i_2} \dots x_{i_n}, \quad (18)$$

where $x_{i_1} < x_{i_2} < \dots < x_{i_n}$ and each x_{i_j} is equal to at least one value in the set $\{1, 2^2, 3^2, \dots, (v-1)^2\}$. For the three lowest values of n the symmetric polynomials are

$$s(v, 0) = 1, \quad s(v, 1) = (v-1)v(2v-1)/6,$$

and

$$s(v, 2) = \frac{(5v+1)}{4 \cdot 6!} (2v-4)_5, \quad (19)$$

while for the four highest values of n they are given by

$$s(v, v-1) = (v-1)!^2, \quad s(v, v-2) = (v-1)!^2 (\zeta(2) - \zeta(2, v)),$$

$$s(v, v-3) = \frac{(v-1)!^2}{2} \left((\zeta(2) - \zeta(2, v))^2 + \zeta(4, v) - \zeta(4) \right),$$

and

$$\begin{aligned} s(v, v-4) = & \frac{(v-1)!^2}{6} \left((\zeta(2) - \zeta(2, v))^3 - 3(\zeta(4) - \zeta(4, v))(\zeta(2) - \zeta(2, v)) \right. \\ & \left. + 2(\zeta(6) - \zeta(6, v)) \right). \end{aligned} \quad (20)$$

The $n = v - \ell$ results have been obtained by beginning with the general form given by (18). For $s(v, v-4)$ this becomes

$$s(v, v-4) = \frac{1}{6} \prod_{i=1}^{v-1} i^2 \sum_{\substack{j_1, j_2, j_3=1 \\ j_1 \neq j_2 \neq j_3}}^{v-1} \frac{1}{j_1^2 j_2^2 j_3^2}. \quad (21)$$

To evaluate this result, the constraint that none of the j_i is equal to another must be removed. This is accomplished by dropping it and eliminating all the possibilities where at least one of j_i is equal to another. Consequently, we must consider subtracting all the cases where two of the indices are equal to another and finally when three indices are equal to one another. The product over i yields $\Gamma(v)^2$. Thus, we arrive at

$$s(v, v-4) = \frac{1}{2} \Gamma(v)^2 \left(\sum_{j_1, j_2, j_3=1}^{v-1} \frac{1}{j_1^2 j_2^2 j_3^2} - 3 \sum_{j_1, j_2=1}^{v-1} \frac{1}{j_1^4 j_2^2} + 2 \sum_{j_1=1}^{v-1} \frac{1}{j_1^6} \right), \quad (22)$$

which yields the result in (20). In a similar manner one finds that

$$\begin{aligned} s(v, v-5) = \frac{1}{24} \Gamma(v)^2 & \left((\zeta(2) - \zeta(2, v))^4 - 6(\zeta(4) - \zeta(4, v))(\zeta(2) - \zeta(2, v))^2 \right. \\ & + 8(\zeta(6) - \zeta(6, v))(\zeta(2) - \zeta(2, v)) + 3(\zeta(4) - \zeta(4, v))^2 \\ & \left. - 6(\zeta(8) - \zeta(8, v)) \right). \end{aligned} \quad (23)$$

See Ref. [11] for more details.

If we introduce the results for $s(v, v-n)$ into (17), then for $n=1$ we obtain

$$\frac{\Gamma(v)}{\Gamma(v+1/2)} = \frac{2}{\sqrt{\pi}} c_{2v, v-1}, \quad (24)$$

which is only valid for $v > 1$. Alternatively, (24) can be expressed as $B(v, 1/2) = 2 c_{2v, v-1}$, where $B(x, y)$ represents the beta function. For $n=2$, (17) yields

$$\sum_{k=1}^{v-1} \frac{1}{k^2} = \frac{\pi^2}{6} - \zeta(2, v) = \frac{2}{3} \frac{c_{2v, v-2}}{c_{2v, v-1}}, \quad (25)$$

where $\zeta(x, y)$ represents the Hurwitz zeta function and $v > 2$. By adopting the same approach for $n=3$ to $n=6$, we derive

$$\sum_{k=1}^{v-1} \frac{1}{k^4} = \frac{\pi^4}{90} - \zeta(4, v) = \frac{4}{9} \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^2 - \frac{4}{15} \frac{c_{2v, v-3}}{c_{2v, v-1}}, \quad (26)$$

$$\sum_{k=1}^{v-1} \frac{1}{k^6} = \frac{\pi^6}{945} - \zeta(6, v) = \frac{4}{105} \frac{c_{2v, v-4}}{c_{2v, v-1}} - \frac{4}{15} \frac{c_{2v, v-3}}{c_{2v, v-1}} \frac{c_{2v, v-2}}{c_{2v, v-1}} + \frac{8}{27} \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^3, \quad (27)$$

$$\begin{aligned} \sum_{k=1}^{v-1} \frac{1}{k^8} = \frac{\pi^8}{9450} - \zeta(8, v) = \frac{8}{14175} & \left(350 \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^4 - 420 \frac{c_{2v, v-3}}{c_{2v, v-1}} \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^2 \right. \\ & \left. + 63 \left(\frac{c_{2v, v-3}}{c_{2v, v-1}} \right)^2 + 60 \frac{c_{2v, v-4}}{c_{2v, v-1}} \frac{c_{2v, v-2}}{c_{2v, v-1}} - 5 \frac{c_{2v, v-5}}{c_{2v, v-1}} \right), \end{aligned} \quad (28)$$

and

$$\begin{aligned} \sum_{k=1}^{v-1} \frac{1}{k^{10}} = \frac{\pi^{10}}{93555} - \zeta(10, v) = \frac{4}{93555} & \left(3080 \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^5 - 4620 \frac{c_{2v, v-3}}{c_{2v, v-1}} \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^3 \right. \\ & + 1386 \left(\frac{c_{2v, v-3}}{c_{2v, v-1}} \right)^2 \frac{c_{2v, v-2}}{c_{2v, v-1}} + 660 \frac{c_{2v, v-4}}{c_{2v, v-1}} \left(\frac{c_{2v, v-2}}{c_{2v, v-1}} \right)^2 - 198 \frac{c_{2v, v-4}}{c_{2v, v-1}} \frac{c_{2v, v-3}}{c_{2v, v-1}} \\ & \left. - 55 \frac{c_{2v, v-5}}{c_{2v, v-1}} \frac{c_{2v, v-2}}{c_{2v, v-1}} + 3 \frac{c_{2v, v-6}}{c_{2v, v-1}} \right). \end{aligned} \quad (29)$$

where $v > 2$ for the first result, $v > 3$ in the second, etc. In principle, this process can be continued for higher powers of the sum on the lhs by determining larger values of ℓ in the symmetric polynomials, $s(v, v - \ell)$. Consequently, we see that integer values of the Hurwitz zeta function for even powers can now be expressed in terms of ratios of the generalized cosecant numbers, which is indeed fascinating in view of the intractability of this famous function. Moreover, in the limit as $v \rightarrow \infty$, we obtain new results for the Riemann zeta function such as

$$\zeta(4) = \frac{\pi^4}{90} = \lim_{v \rightarrow \infty} \left\{ \frac{4}{9} \left(\frac{c_{2v,v-2}}{c_{2v,v-1}} \right)^2 - \frac{4}{15} \frac{c_{2v,v-3}}{c_{2v,v-1}} \right\}. \quad (30)$$

Unfortunately, we cannot introduce (16) in the above result because $|\rho/k|$ is approximately equal to 2 since k is equal to either $v - 1$ and $v - 2$, whereas we have observed in Table 3 that ρ or $2v$ needs to be very much greater than k in order that $\beta(\rho, k)$ remains close to unity. Therefore, in order to study the $v \rightarrow \infty$ limit far more values of the generalized cosecant numbers than those in Table 2 are required, which is the aim of Refs. [9]. By introducing the values of ρ and k occurring in the (25)-(27) into these results, one may gain the necessary insight into the large v behaviour.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF MELBOURNE, VICTORIA 3010, AUSTRALIA

E-mail address: vkowa@unimelb.edu.au, vkowal@netspace.net.au